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Particle with spin 1 in a magnetic field on the spherical plane S_2

Abstract

There are constructed exact solutions of the quantum-mechanical equation for a spin $S=1$ particle in 2-dimensional Riemannian space of constant positive curvature, spherical plane, in presence of an external magnetic field, analogue of the homogeneous magnetic field in the Minkowski space. A generalized formula for energy levels describing quantization of the motion of the vector particle in magnetic field on the 2-dimensional space S_2 has been found, nonrelativistic and relativistic equations have been solved.

1 Introduction

The quantization of a quantum-mechanical particle in the homogeneous magnetic field belongs to classical problems in physics [1, 2, 3, 4]. In 1985 – 2010, a more general problem in a curved Riemannian background, hyperbolic and spherical planes, was extensively studied [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], providing us with a new system having intriguing dynamics and symmetry, both on classical and quantum levels.

Extension to 3-dimensional hyperbolic and spherical spaces was performed recently. In [25, 26, 27], exact solutions for a scalar particle in extended problem, particle in external magnetic field on the background of Lobachevsky H_3 and Riemann S_3 spatial geometries were found. A corresponding system in the frames of classical mechanics was examined in [28, 29, 30]. In the present paper, we consider a quantum-mechanical problem a particle with spin 1/2 described by the Dirac equation in 3-dimensional Lobachevsky and Riemann space models in presence of the external magnetic field.

In the present paper, we will construct exact solutions for a vector particle described by 10-dimensional Duffin–Kemmer equation in external magnetic field on the background of 2-dimensional spherical space S_2.

10-dimensional Duffin–Kemmer equation for a vector particle in a curved space-time has the form [31]

$$\{\beta^{c}[\, i\hbar\, (\, e^{\beta}_{(c)}\partial_{\beta} + \frac{1}{2}J^{ab}\gamma_{abc}\,) \, + \, \frac{e}{c}A_{c}\,] \, - \, mc\}\Psi = 0 \, , \tag{1.1}$$

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where γ_{abc} stands for Ricci rotation coefficiemts, $A_a = e^{\beta}_{(a)} A_\beta$ represent tetrad components of electromagnetic 4-vector A_β; $J^{ab} = \beta^a \beta^b - \beta^b \beta^a$ are generators of 10-dimensional representation of the Lorentz group. For shortness, we use notation $e/c\hbar \Longrightarrow e,\ mc/\hbar \Longrightarrow M$.

In the space S_3 we will use the system of cylindric coordinates [32]

$$dS^2 = c^2 dt^2 - \rho^2 [\cos^2 z (dr^2 + \sin^2 r \, d\phi^2) + dz^2],$$
$$z \in [-\pi/2, +\pi/2],\ r \in [0, +\pi],\ \phi \in [0, 2\pi]. \tag{1.2}$$

Generalized expression for electromagnetic potential for an homogeneous magnetic field in the curved model S_3 is given as follows

$$A_\phi = -2B \sin^2 \frac{r}{2} = B\ (\cos r - 1)\ . \tag{1.3}$$

We will consider the above equation in presence of the field in the model S_3. Corresponding to cylindric coordinates $x^\alpha = (t, r, \phi, z)$ a tetrad can be chosen as

$$e^{\beta}_{(a)}(x) = \left| \begin{array}{llll} 1 & 0 & 0 & 0 \\ 0 & \cos^{-1} z & 0 & 0 \\ 0 & 0 & \cos^{-1} z\ \sin^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| . \tag{1.4}$$

Eq. (1.1) takes the form

$$\left\{ i\beta^0 \frac{\partial}{\partial t} + \frac{1}{\cos z} \left(i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial_\phi + eB(\cos r - 1) + iJ^{12} \cos r}{\sin r} \right) \right.$$
$$\left. + i\beta^3 \frac{\partial}{\partial z} + i \frac{\sin z}{\cos z}\ (\beta^1 J^{13} + \beta^2 J^{23}) - M \right\} \Psi = 0\ , \tag{1.5}$$

To separate the variables in eq. (1.5), we are to employ an explicit form of the Duffin–Kemmer matrices β^a; it will be most convenient to use so called cyclic representation [34], where the generator J^{12} is of diagonal form (we specify matrices by blocks in accordance with ($1-3-3-3$)-splitting)

$$\beta^0 = \left| \begin{array}{rrrr} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right| , \qquad \beta^i = \left| \begin{array}{rrrr} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & \tau_i \\ -e_i^+ & 0 & 0 & 0 \\ 0 & -\tau_i & 0 & 0 \end{array} \right| , \tag{1.6}$$

where $e_i,\ e_i^t,\ \tau_i$ denote

$$e_1 = \frac{1}{\sqrt{2}}(-i,\ 0,\ i)\ , \qquad e_2 = \frac{1}{\sqrt{2}}(1,\ 0,\ 1)\ , \qquad e_3 = (0, i, 0)\ ,$$

$$\tau_1 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right| , \qquad \tau_2 = \frac{1}{\sqrt{2}} \left| \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{array} \right| , \qquad \tau_3 = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right| = s_3\ .$$

$$(1.7)$$

The generator J^{12} explicitly reads

$$J^{12} = \beta^1\beta^2 - \beta^2\beta^1 = -i \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & \tau_3 & 0 & 0 \\ 0 & 0 & \tau_3 & 0 \\ 0 & 0 & 0 & \tau_3 \end{array} \right| = -iS_3. \tag{1.8}$$

2 Restriction to 2-dimensional model

Let us restrict ourselves to 2-dimensional case, spherical space S_2 (formally it is sufficient in eq. (1.5) to remove dependence on the variable z fixing its value by $z = 0$)

$$\left[i\beta^0 \frac{\partial}{\partial t} + i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial_\phi + eB(\cos r - 1) + iJ^{12}\cos r}{\sin r} - M \right] \Psi = 0 \ . \tag{2.1}$$

With the use of substitution

$$\Psi = e^{-i\epsilon t} e^{im\phi} \left| \begin{array}{c} \Phi_0(r) \\ \vec{\Phi}(r) \\ \vec{E}(r) \\ \vec{H}(r) \end{array} \right| , \tag{2.2}$$

eq. (2.1) assi=umes the form (introducing notation $m + B(1 - \cos r) = \nu(r)$)

$$\left[\epsilon \, \beta^0 + i\beta^1 \, \frac{\partial}{\partial r} - \beta^2 \, \frac{\nu(r) - \cos r \; S_3}{\sin r} - M \right] \left| \begin{array}{c} \Phi_0(r) \\ \vec{\Phi}(r) \\ \vec{E}(r) \\ \vec{H}(r) \end{array} \right| = 0 \ . \tag{2.3}$$

After separation of the variables and using the notation

$$\frac{1}{\sqrt{2}}(\frac{\partial}{\partial r} + \frac{\nu - \cos r}{\sin r}) = \hat{a}_- \ , \quad \frac{1}{\sqrt{2}}(\frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r}) = \hat{a}_+ \ , \quad \frac{1}{\sqrt{2}}(\frac{\partial}{\partial r} + \frac{\nu}{\sin r}) = \hat{a} \ ,$$

$$\frac{1}{\sqrt{2}}(-\frac{\partial}{\partial r} + \frac{\nu - \cos r}{\sin r}) = \hat{b}_- \ , \quad \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial r} + \frac{\nu + \cos r}{\sin r}) = \hat{b}_+ \ , \quad \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial r} + \frac{\nu}{\sin r}) = \hat{b}$$

we arrive at the radial system

$$-\hat{b}_- E_1 - \hat{a}_+ E_3 = M \ \Phi_0 \ , \qquad -i\hat{b}_- H_1 + i\hat{a}_+ \ H_3 + i\epsilon \ E_2 = M \ \Phi_2 \ ,$$

$$i\hat{a} H_2 + i\epsilon \ E_1 = M \ \Phi_1 \ , \qquad -i\hat{b} H_2 + i\epsilon \ E_3 = M \ \Phi_3 \ , \tag{2.4}$$

$$\hat{a}\Phi_0 - i\epsilon \ \Phi_1 = M \ E_1 \ , \qquad -i\hat{a}\Phi_2 = M \ H_1 \ , \qquad \hat{b}\Phi_0 - i\epsilon \ \Phi_3 = M \ E_3 \ ,$$

$$i\hat{b}\Phi_2 = M \ H_3 \ . \qquad - i\epsilon\Phi_2 = M \ E_2 \ , \qquad i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3 = M \ H_2 \ , \tag{2.5}$$

3 Nonrelativistic approximation

Excluding non-dynamical variables Φ_0, H_1, H_2, H_3 with the help of equations

$$-\hat{b}_- E_1 - \hat{a}_+ E_3 = M\ \Phi_0\ , \qquad -i\ \hat{a}\Phi_2 = M\ H_1\ ,$$

$$i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3 = M\ H_2\ , \qquad i\ \hat{b}\Phi_2 = M\ H_3\ . \tag{3.1}$$

we get 6 equations (grouping them in pairs)

$$i\hat{a}\ (i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3) + i\epsilon\ M E_1 = M^2 \Phi_1\ ,$$

$$\hat{a}\ (-\hat{b}_- E_1 - \hat{a}_+ E_3 - i\epsilon\ M\Phi_1 = M^2 e_1\ ,$$

$$(3.2a)$$

$$-i\hat{b}_-\ (-i\ \hat{a}\ \Phi_2) + i\hat{a}_+ (i\ \hat{b}\ \Phi_2) + i\epsilon\ M E_2 = M^2 \Phi_2\ ,$$

$$-i\epsilon\ M\Phi_2 = M^2 E_2\ ,$$

$$(3.2b)$$

$$-i\ \hat{b}\ (i\hat{b}_- \Phi_1 - i\hat{a}_+ \Phi_3) + i\epsilon\ M E_3 = M^2 \Phi_3\ ,$$

$$\hat{b}\ (-\hat{b}_- E_1 - \hat{a}_+ E_3) - i\epsilon\ M\Phi_3 = M^2 E_3\ ,$$

$$(3.2c)$$

Now we introduce big and small constituents

$$\Phi_1 = \Psi_1 + \psi_1\ , \qquad \Phi_2 = \Psi_2 + \psi_2\ , \qquad \Phi_3 = \Psi_3 + \psi_3\ ,$$

$$iE_1 = \Psi_1 - \psi_1\ , \qquad iE_2 = \Psi_2 - \psi_2\ , \qquad iE_3 = \Psi_3 - \psi_3\ ,$$

besides we should separate the rest energy by formal change $\epsilon \Longrightarrow \epsilon + M$; summing and subtracting equation within each pair (3.2) and ignoring small constituents ψ_i we arrive at three equations for big components

$$\Big(-2\ \hat{a}\hat{b}_- + 2\epsilon M\Big)\ \psi_1 = 0\ ,$$

$$\Big(-\hat{b}_- \hat{a} - \hat{a}_+ \hat{b} + 2\epsilon M\Big)\ \psi_2 = 0\ ,$$

$$\Big(-2\hat{b}\hat{a}_+ + 2\epsilon M\Big)\ \psi_3 = 0\ . \tag{3.3}$$

It is a needed Pauli-like system for the spin 1 particle.

Allowing for $\nu(r) = m + B\,(1-\cos r)$, from (3.3) we get radial equations in the form

$$\Big[\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\,\frac{d}{dr} - B - \frac{1 - 2\,[m + B\,(1-\cos r)]\ \cos r}{\sin^2 r} -$$

$$- \frac{[m + B\,(1-\cos r)]^2}{\sin^2 r} + 2\,\epsilon\, M\Big]\ \psi_1 = 0\,,$$

$$(3.4a)$$

$$\left[\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} - \frac{[m + B\,(1-\cos r)]^2}{\sin^2 r} + 2\,\epsilon\,M \right] \psi_2 = 0\,,$$

$$(3.4b)$$

$$\left[\frac{d^2}{dr^2} + \frac{\cos r}{\sin r}\frac{d}{dr} + B - \frac{1 + 2\,[m + B\,(1-\cos r)]\,\cos r}{\sin^2 r} - \right.$$
$$\left. - \frac{[m + B\,(1-\cos r)]^2}{\sin^2 r} + 2\,\epsilon\,M \right] \psi_3 = 0\,.$$

$$(3.4c)$$

The first and the third equations are symmetric with respect to formal change $m \Longrightarrow -m\,,\; B \Longrightarrow -B$.

Let us consider eq. (3.4a). In the new variable $1 - \cos r = 2\,y$; and with the use of a substitution $B_1 = y^{C_1}(1-y)^{A_1} f_1$ the differential equation assumes the form

$$y\,(1-y)\,\frac{d^2 B_1}{dy^2} + [2\,C_1 + 1 - (2\,A_1 + 2\,C_1 + 2)\,y]\,\frac{dB_1}{dy}$$
$$+ \big[B^2 + B + 2\,\epsilon\,M - (A_1 + C_1)\,(A_1 + C_1 + 1)$$
$$+ \frac{1}{4}\,\frac{4\,A_1^2 - (2\,B + m + 1)^2}{1-y} + \frac{1}{4}\,\frac{4\,C_1^2 - (m-1)^2}{y} \bigg] B_1 = 0\,. \tag{3.5}$$

At A_1, C_1 taken according to

$$A_1 = +\frac{1}{2}\mid 2B + m + 1 \mid\,, \qquad C_1 = +\frac{1}{2}\mid m - 1 \mid\,, \tag{3.6}$$

eq. (3.5) is recognized as a hypergeometric equation [33] with parameters

$$\alpha_1 = A_1 + C_1 + \frac{1}{2} - \sqrt{B^2 + B + 2\,\epsilon\,M + \frac{1}{4}}\,,$$

$$\beta_1 = A_1 + C_1 + \frac{1}{2} + \sqrt{B^2 + B + 2\,\epsilon\,M + \frac{1}{4}}\,,$$

$$\gamma_1 = 2\,C_1 + 1\,, \qquad B_1 = y^{C_1}\,(1-y)^{A_1}\,F\,(\alpha_1,\,\beta_1,\,\gamma_1;\,y). \tag{3.7}$$

To get polynomials we need to impose restriction $\alpha_1 = -n$, from this it follows

$$\psi_1 = y^{C_1}(1-y)^{A_1}\,F(\alpha_1, \beta_1, \gamma_1;\, y)\,,$$

$$\sqrt{B^2 + B + 2\,\epsilon\,M + \frac{1}{4}} = n + \frac{1}{2} + \frac{\mid 2B + m + 1 \mid + \mid m - 1 \mid}{2}\,. \tag{3.8}$$

In similar manner, we construct solution of eq. (3.4b)

$$\psi_2 = y^{C_2}(1-y)^{A_2}\,F(\alpha_2, \beta_2, \gamma_2;\, y)\,,$$

$$A_2 = \pm\frac{1}{2}\,(2\,B + m)\,, \;\; C_2 = \pm\,\frac{m}{2}\,, \;\; \gamma_2 = 2\,C_2 + 1\,,$$

$$\alpha_2 = A_2 + C_2 + \frac{1}{2} - \sqrt{B^2 + 2\,\epsilon\,M + \frac{1}{4}}\,,$$

$$\beta_2 = A_2 + C_2 + \frac{1}{2} + \sqrt{B^2 + 2\,\epsilon\,M + \frac{1}{4}}\,; \tag{3.9}$$

from quantization condition $\alpha_2 = -n$ it follows

$$\sqrt{B^2 + 2\,\epsilon\,M + \frac{1}{4}} = n + \frac{1}{2} + \frac{\mid 2B + m \mid + \mid m \mid}{2}\;. \tag{3.10}$$

Finally, we construct solutions for eq. (3.4c):

$$\psi_3 = y^{C_3}(1-y)^{A_3}\,F(\alpha_3, \beta_3, \gamma_3;\, y)\,,$$

$$A_3 = \pm\frac{1}{2}\,(2\,B + m - 1)\,, \quad C_3 = \pm\frac{1}{2}\,(m+1)\,, \quad \gamma_3 = 2\,C_3 + 1\,,$$

$$\alpha_3 = A_3 + C_3 + \frac{1}{2} - \sqrt{B^2 - B + 2\,\epsilon\,M + \frac{1}{4}}\,,$$

$$\beta_3 = A_3 + C_3 + \frac{1}{2} + \sqrt{B^2 - B + 2\,\epsilon\,M + \frac{1}{4}}\,; \tag{3.11}$$

and requiring $\alpha_3 = -n$ we obtain

$$\sqrt{B^2 - B + 2\,\epsilon\,M + \frac{1}{4}} = n + \frac{1}{2} + \frac{\mid 2B + m - 1 \mid + \mid m + 1 \mid}{2}\;. \tag{3.12}$$

4 Solution of radial equations in relativistic case

Let start with eqs. (2.4)–(2.5). Excluding six components E_i, H_i with the help of (2.5), we derive four second order equations for Φ_a:

$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0\;,$$

$$(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon(\hat{b}_-\Phi_1 + \hat{a}_+\Phi_3) = 0\;,$$

$$(-\hat{a}\hat{b}_- + \epsilon^2 - M^2)\Phi_1 + \hat{a}\hat{a}_+\Phi_3 + i\epsilon\hat{a}\Phi_0 = 0\;,$$

$$(-\hat{b}\hat{a}_+ + \epsilon^2 - M^2)\Phi_3 + \hat{b}\hat{b}_-\Phi_1 + i\epsilon\hat{b}\Phi_0 = 0\;. \tag{4.1}$$

Once, it should be noted existence of a simple solution of the system

$$\Phi_0 = 0\;, \qquad \Phi_1 = 0\;, \qquad \Phi_3 = 0\;, \qquad (-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0\;, \tag{4.2a}$$

at this from (2.5) it follows

$$E_1 = 0\;, \qquad E_2 = -i\epsilon M^{-1}\Phi_2\;, \qquad E_3 = 0\;,$$

$$H_1 = -iM^{-1}\hat{a}\;\Phi_2\;, \qquad H_2 = 0\;, \qquad H_3 = iM^{-1}\hat{b}\;\Phi_2\;. \tag{4.2b}$$

Lets us turn to (4.1) and act on the third equation from the left by operator $\hat{b}_-$, and on the forth equation by operator $\hat{a}_+$. Thus, introducing the notation

$$\hat{b}_-\Phi_1 = Z_1 \; , \qquad \hat{a}_+\Phi_3 = Z_3 \; ,$$

instead of (4.1) we obtain

$$\begin{gathered}
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0 \; , \\
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon(Z_1 + Z_3) = 0 \; , \\
(-\hat{b}_-\hat{a} + \epsilon^2 - M^2)Z_1 + \hat{b}_-\hat{a}Z_3 + i\epsilon\hat{b}_-\hat{a}\Phi_0 = 0 \; , \\
(-\hat{a}_+\hat{b} + \epsilon^2 - M^2)Z_3 + \hat{a}_+\hat{b}Z_1 + i\epsilon\hat{a}_+\hat{b}\Phi_0 = 0 \; .
\end{gathered} \tag{4.3}$$

Instead of Z_1, Z_3, let us use new variables

$$Z_1 = \frac{f+g}{2} \; , \qquad Z_3 = \frac{f-g}{2} \; ,$$

$$Z_1 + Z_3 = f \; , \qquad Z_1 - Z_3 = g \; ;$$

then the system assumes the form

$$\begin{gathered}
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0 \; , \\
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon \; f = 0 \; , \\
-\hat{b}_-\hat{a}\frac{f+g}{2} + (\epsilon^2 - M^2)\frac{f+g}{2} + \hat{b}_-\hat{a}\frac{f-g}{2} + i\epsilon\hat{b}_-\hat{a}\Phi_0 = 0 \; , \\
-\hat{a}_+\hat{b}\frac{f-g}{2} + (\epsilon^2 - M^2)\frac{f-g}{2} + \hat{a}_+\hat{b}\frac{f+g}{2} + i\epsilon\hat{a}_+\hat{b}\Phi_0 = 0 \; .
\end{gathered} \tag{4.4}$$

Summing and subtract ing equations 3 and , we get

$$\begin{gathered}
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} + \epsilon^2 - M^2)\Phi_2 = 0 \; , \\
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} - M^2)\Phi_0 + i\epsilon \; f = 0 \; , \\
(-\hat{b}_-\hat{a} + \hat{a}_+\hat{b})g + (\epsilon^2 - M^2)f + i\epsilon(\hat{b}_-\hat{a} + \hat{a}_+\hat{b}) \; \Phi_0 = 0 \; , \\
(-\hat{b}_-\hat{a} - \hat{a}_+\hat{b})g + (\epsilon^2 - M^2)g + i\epsilon(\hat{b}_-\hat{a} - \hat{a}_+\hat{b})\Phi_0 = 0 \; ,
\end{gathered} \tag{4.5}$$

Taking into account identities

$$-\hat{b}_-\hat{a} - \hat{a}_+\hat{b} = \Delta_2, \qquad -\hat{b}_-\hat{a} + \hat{a}_+\hat{b} = 2B \tag{4.6}$$

eqs. (4.5) reduce to the form

$$(\Delta_2 + \epsilon^2 - M^2) \; \Phi_2 = 0 \; , \tag{4.7}$$

$$\begin{gathered}
(\Delta_2 - M^2) \; \Phi_0 + i\epsilon \; f = 0 \; , \\
2B \; g + (\epsilon^2 - M^2)f - i\epsilon\Delta_2 \; \Phi_0 = 0 \; , \\
\Delta_2 g + (\epsilon^2 - M^2)g - 2i\epsilon B \; \Phi_0 = 0 \; ,
\end{gathered} \tag{4.8}$$

From the second equation, with the use of expression for $\Delta_2 \Phi_0$ according to the first equation, we derive linear relation between three functions

$$2B\; g - M^2 f - i\epsilon M^2 \Phi_0 = 0\; . \tag{4.9}$$

With the help of (4.9), let us exclude f

$$f = \frac{2B}{M^2}\, g - i\epsilon \Phi_0$$

from equations 2 and 3:

$$(\Delta_2 + \epsilon^2 - M^2)\; g = 2i\epsilon B\; \Phi_0\; ,$$

$$(\Delta_2 + \epsilon^2 - M^2)\; \Phi_0 = -\frac{2i\epsilon B}{M^2}\; g. \tag{4.10}$$

With notation $\gamma = \epsilon^2/M^2$, the system can be written in a matrix form

$$(\Delta_2 + \epsilon^2 - M^2) \left| \begin{array}{c} g \\ \epsilon\; \Phi_0 \end{array} \right| = \left| \begin{array}{cc} 0 & 2iB \\ -2iB\gamma & 0 \end{array} \right| \left| \begin{array}{c} g \\ \epsilon\; \Phi_0 \end{array} \right| \tag{4.11}$$

or symbolically as

$$\Delta f = A f \qquad \Delta f' = SAS^{-1}\; f'\; , \qquad f' = Sf\; .$$

It remains to find a transformation reducing the matrix A to a diagonal form

$$SAS^{-1} = \left| \begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right| , \qquad S = \left| \begin{array}{cc} a & d \\ c & b \end{array} \right| \; ;$$

the problem is equivalent to the linear system

$$-\lambda_1\; a - 2i\gamma B\; d = 0\; , \qquad 2iB\; a - \lambda_1\; d = 0\; ;$$

$$-\lambda_2\; c - 2i\gamma B\; b = 0\; , \qquad 2iB\; c - \lambda_2\; b = 0\; .$$

Its solutions can be chosen in the form

$$\lambda_1 = +\frac{2\epsilon B}{M}\; , \qquad \lambda_2 = -\frac{2\epsilon B}{M}\; ,\; S = \left| \begin{array}{cc} \epsilon & +iM \\ \epsilon & -iM \end{array} \right| ,$$

$$S^{-1} = \frac{1}{-2i\epsilon M} \left| \begin{array}{cc} -iM & -iM \\ -\epsilon & \epsilon \end{array} \right| . \tag{4.12}$$

New (primed) function satisfy the following equations

$$1) \qquad \left(\Delta_2 + \epsilon^2 - M^2 - \frac{2\epsilon B}{M} \right)\; g' = 0\; , \tag{4.13a}$$

$$2) \qquad \left(\Delta_2 + \epsilon^2 - M^2 + \frac{2\epsilon B}{M} \right)\; \Phi_0' = 0\; . \tag{4.13b}$$

They independent from each other. There exist two linearly independent ones

$$1) \qquad g' \neq 0, \qquad \Phi_0' = 0\; , \qquad 2) \qquad g' = 0, \qquad \Phi_0' \neq 0\; . \tag{4.14}$$

The initial functions for these two cases assume respectively the form

$$1) \qquad g = \frac{1}{2\epsilon} g' \; , \qquad \epsilon \Phi_0 = \frac{1}{2iM} g' \; ;$$

$$2) \qquad g = \frac{1}{2i\epsilon} \epsilon \Phi_0' \; , \qquad \epsilon \Phi_0 = -\frac{1}{2iM} \epsilon \Phi_0' \; .$$

In each case, eqs. (4.10) have the same form coinciding with (4.13a) and (4.13b) respectively. To obtain explicit solutions for these differential equation, we need not any additional calculations, instead it suffices to perform simple formal changes as pointed below

$$2\,\epsilon\, M \qquad \Longrightarrow \qquad \left\{ \begin{array}{l} (\epsilon^2 - M^2 - \frac{2\epsilon B}{M}) \\ (\epsilon^2 - M^2) \\ (\epsilon^2 - M^2 + \frac{2\epsilon B}{M}) \end{array} \right. \tag{4.15}$$

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